Available online at www.sciencedirect.com
SCIENCE DIRECT•
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 50 (2005) 1639–1646
www.elsevier.com/locate/camwa

A Mixed Variational Formulation for the Navier-Stokes Problem with Hyper-Dissipation

H. MANOUZI
Département de Mathématiques et statistique
Université Laval, Québec, Canada, G1K7P4

(Received June 2003; revised and accepted June 2005)

Abstract—A mixed variational formulation is used to solve the stationary Navier-Stokes equations with hyper dissipation. In this formulation, the laplacian of the velocity, the velocity and the pressure are the most relevant unknowns. For the linear case, the existence and uniqueness results for this mixed formulation are proved. Then, numerical results are presented for the nonlinear case. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Mixed finite-element method, Navier-Stokes problem, Hyper-dissipation.

1. INTRODUCTION

In this paper, we consider the Navier-Stokes problem with hyper-dissipation,

$$
\begin{aligned}
\epsilon\Delta^2\mathbf{u} + [-\nu\Delta\mathbf{u} + \mathbf{u}.\nabla\mathbf{u}] + \nabla p &= \mathbf{f}, && \text{in } \Omega,\\
\nabla.\mathbf{u} &= 0, && \text{in } \Omega,\\
\mathbf{u} &= 0, && \text{on } \Gamma,\\
\frac{\partial\mathbf{u}}{\partial\mathbf{n}} &= 0, && \text{on } \Gamma,
\end{aligned}
\tag{1.1}
$$

where $\mathbf{u}$ stands for the velocity field, p, the pressure field, $\mathbf{f}$, a given body force, and ν, the kinematic viscosity. We will assume that ϵ and ν are positive constants and Ω is a two- or three-dimensional Lipschitz domain with the boundary Γ.

Mathematical model for such fluid motion play an important role in theoretical and computational studies of bipolar fluids [1–5], and in the regularized Navier-Stokes equations [6–9].

The Navier-Stokes equations are the starting point for most numerical simulations of turbulent fluids which are characterised by higher Reynolds numbers. The number of degrees of freedom required for the direct simulation of such fluids increases with the complexity of the flow. To overcome such problems, one can consider modifications to the equations which allow the computation of more turbulent flows. For example, in such models the operator $-\Delta$, responsible for dissipating energy from the system, is replaced by a higher-order dissipation mechanism which

0898-1221/05/$ - see front matter Typeset by $\mathcal{AMS}$-TeX
doi:10.1016/j.camwa.2005.06.007

damps the higher wave numbers more selectively. The operator $(-\Delta)^\alpha, \alpha > 1$ is a typical choice. The presence of hyper-dissipation results in a decrease of the energy of the linear term, hence, an increase in the effective Reynolds number.

Hyper-viscosity is introduced in [6–8] to demonstrate global unique solvability of the Navier-Stokes equations in three dimensions.

Another reason for introducing a hyper-viscosity comes from the fact that the Navier-Stokes equations are based on the assumption on Newtonian flows. For a non-Newtonian fluids, one may introduce hyper-viscosity as in the paper of [1,2].

The object of this paper is to introduce a mixed formulation to (1.1) for the linear case. This allows us to reduce the order of the problem and then makes the numerical approximation more easier to handle.

Our focus is the mixed formulation that relaxed the regularity requirement on $\mathbf{u}$ and converted the essential boundary condition $\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0$ into a natural one. Both of these features facilitate finite-element approximations: it is easier to construct H^1 finite-element spaces than H^2 finite-element spaces; it is easier to construct finite-element spaces without boundary conditions than with boundary conditions.

In the case of the nonlinear problem, we present only the results of some numerical experiments.

2. THE PRIMAL VARIATIONAL FORMULATION: THE LINEAR CASE

For our analysis, we shall consider only the following linear problem,

$$\begin{aligned} \epsilon\Delta^2\mathbf{u} + \nabla p &= \mathbf{f}, && \text{in } \Omega, \\ \nabla.\mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u} &= 0, && \text{on } \Gamma, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}} &= 0, && \text{on } \Gamma, \end{aligned} \tag{2.1}$$

The nonlinear case can be analyzed by using the standard Galerkin fixed-point theory [7,9].

We introduce some notation that will be used in the sequel.

$$\begin{aligned} L_0^2(\Omega) &= \{q \in L^2 : \int_\Omega q\, d\Omega = 0\}, \\ \mathbf{H}_0^2(\Omega) &= \{\mathbf{v} \in \mathbf{H}^2 : \mathbf{v} = 0,\ \frac{\partial \mathbf{v}}{\partial \mathbf{n}} = 0,\ \text{on } \Gamma\}, \\ V &= \{\mathbf{v} \in \mathbf{H}_0^2(\Omega),\ \operatorname{div}\mathbf{v} = 0\}. \end{aligned}$$

These Sobolev spaces are equipped with their natural Hilbertian norms. For details concerning these spaces consult [10,11].

The duality pairing between a Sobolev space X and its dual space X' is denoted by $\langle .,. \rangle_{X',X}$.

We have the following existence and uniqueness results [12].

THEOREM 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a boundary Γ.*

1. *Assume that Γ is of class $C^{1,1}$ and that $\mathbf{f} \in \mathbf{H}^{-2}$, then problem (2.1) has at least one solution $\mathbf{u} \in \mathbf{H}_0^2(\Omega)$, $p \in \mathbf{H}^{-1}$.*
2. *Moreover, if Γ is of class $C^{k+3,1}$ for an integer $k \geq 0$ and that $\mathbf{f} \in \mathbf{H}^k$, then $\mathbf{u} \in \mathbf{H}^{k+4}$, $p \in \mathbf{H}^k \cap L_0^2(\Omega)$ and there exists a constant $C > 0$, such that*

$$\|\mathbf{u}\|_{k+4,\Omega} + \|p\|_{k,\Omega} \leq C\|\mathbf{f}\|_{k,\Omega}.$$

To approximate problem (2.1) by a conforming finite-element method for example, we have to construct a finite-dimensional subspace of $\mathbf{H}^2$. These functions must be in $(C^1(\Omega))^2$. To overcome this regularity condition, it is preferable to use a mixed formulation.

3. A MIXED VARIATIONAL FORMULATION

Our mixed formulation is based on the introduction of $\mathbf{w} = \Delta\mathbf{u}$ as a new unknown. With this new unknown, problem (2.1) is decomposed into a second-order system as follows,

$$\begin{aligned} \mathbf{w} - \Delta\mathbf{u} &= 0, && \text{in } \Omega, \\ \Delta\mathbf{w} + \frac{1}{\epsilon}\nabla p &= \frac{1}{\epsilon}\mathbf{f}, && \text{in } \Omega, \\ \nabla.\mathbf{u} &= 0, && \text{in } \Omega, \\ \mathbf{u} &= 0, && \text{on } \Gamma, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}} &= 0, && \text{on } \Gamma. \end{aligned}$$

We introduce the following spaces,

$$X := \mathbf{H}_0^1(\Omega), \qquad Y := L_0^2(\Omega),$$

and the divergence operator D is defined by

$$D : X \longrightarrow Y, \qquad D\mathbf{v} = \frac{1}{\epsilon}\operatorname{div}\mathbf{v},$$

and we set

$$H := \operatorname{Ker}D = \{\mathbf{v} \in X,\ \operatorname{div}\mathbf{v} = 0\}.$$

We have the inclusions,

$$H \subset X \subset X' \subset H', \text{ with continuous imbeddings,}$$

and we set

$$W := \left\{\mathbf{w} \in \mathbf{L}^2(\Omega),\ \Delta\mathbf{w} \in H'\right\}.$$

The Hilbert spaces W, X are equipped with their natural norms,

$$\|\mathbf{w}\|_W^2 = \|\mathbf{w}\|_{0,\Omega}^2 + \|\Delta\mathbf{w}\|_{H'}^2, \quad \|\mathbf{v}\|_X = \|\operatorname{grad}\mathbf{v}\|_{0,\Omega}, \qquad \mathbf{w} \in W, \quad \mathbf{v} \in X.$$

Next, we introduce the operator B defined by

$$B : W \longrightarrow H', \qquad B\mathbf{z} = \Delta\mathbf{z}, \qquad \mathbf{z} \in W,$$

and its dual transpose operator $B' : H \longrightarrow W'$ by the relation,

$$\langle B\mathbf{z}, \mathbf{v}\rangle_{H',H} = \langle \mathbf{z}, B'\mathbf{v}\rangle_{W,W'},$$

and we introduce the mass operator $A : W \longrightarrow W'$ such that $\langle A\mathbf{w}, \mathbf{z}\rangle = \int_\Omega \mathbf{w}.\mathbf{z}\,dx$. So that the equation $\mathbf{w} = \Delta\mathbf{u}$ becomes in a weak version,

$$A\mathbf{w} = B'\mathbf{u}, \qquad \text{in } W', \quad \mathbf{w} \in W, \quad \mathbf{u} \in X.$$

With these notations, the biharmonic Stokes problem (2.1) becomes

$$\begin{aligned} &\text{find } \mathbf{w} \in W,\ \mathbf{u} \in H, \text{ such that,} \\ &A\mathbf{w} = B'\mathbf{u}, \qquad \text{in } W', \\ &B\mathbf{w} = \frac{1}{\epsilon}\mathbf{f}, \qquad \text{in } H', \end{aligned} \tag{3.2}$$

or in the variational formulation,

$$\begin{aligned} &\text{find } \mathbf{w} \in W,\ \mathbf{u} \in H \text{ such that,} \\ &\langle A\mathbf{w}, \mathbf{z}\rangle_{W',W} - \langle B'\mathbf{u}, \mathbf{z}\rangle_{W',W} = 0, \qquad \forall\,\mathbf{z} \in W, \\ &\langle B\mathbf{w}, \mathbf{v}\rangle_{H',H} = \frac{1}{\epsilon}\langle \mathbf{f}, \mathbf{v}\rangle_{H',H}, \qquad \forall\,\mathbf{v} \in H, \end{aligned}$$

We note here that the Neumann boundary condition for $\mathbf{u}$ is implicitly satisfied via this weak formulation.

We list the properties implied by the introduction of the operators A and B.

LEMMA 3.1. *We have the following characterization,*

$$V := \{\mathbf{w} \in W,\ \langle B\mathbf{w}, \mathbf{v}\rangle_{H',H} = 0,\ \forall\, \mathbf{v} \in H\}$$
$$= \{\mathbf{w} \in W,\ \Delta \mathbf{w} = 0 \ \text{in } H'\}.$$

As a consequence of Lemma 3.1, we have the following.

LEMMA 3.2. V-ELLIPTICITY. *The operator A is elliptic on the space V, i.e., there exists a constant $\alpha > 0$, such that*

$$\langle A\mathbf{w}, \mathbf{w}\rangle_{W',W} \geq \alpha \|\mathbf{w}\|_W^2, \qquad \forall\, \mathbf{w} \in V.$$

PROOF. We have $\|\mathbf{w}\|_W = \|\mathbf{w}\|_{0,\Omega}$, for all $\mathbf{w} \in V$. ∎

LEMMA 3.3. INF-SUP CONDITION. *There exists a constant $\gamma > 0$, such that*

$$\sup_{\mathbf{w}\in W} \frac{\langle \mathbf{v}, B\mathbf{w}\rangle_{H,H'}}{\|\mathbf{w}\|_W} \geq \gamma \|\mathbf{v}\|_X, \qquad \forall\, \mathbf{v} \in H.$$

PROOF. Since $H \subset W$, we have, for all $\mathbf{v} \in H$,

$$\begin{aligned}
\sup_{\mathbf{w}\in W} \frac{\langle \mathbf{v}, B\mathbf{w}\rangle_{H,H'}}{\|\mathbf{w}\|_W} &\geq \sup_{\mathbf{w}\in H} \frac{\langle \mathbf{v}, B\mathbf{w}\rangle_{H,H'}}{\|\mathbf{w}\|_W} \\
&= \sup_{\mathbf{w}\in H} \frac{(\operatorname{grad}\mathbf{w}, \operatorname{grad}\mathbf{v})_{0,\Omega}}{\|\mathbf{w}\|_X} \\
&\geq \frac{(\operatorname{grad}\mathbf{v}, \operatorname{grad}\mathbf{v})_{0,\Omega}}{\|\mathbf{v}\|_X} \\
&\geq C\|\mathbf{v}\|_X.
\end{aligned}$$

∎

Let us state the main result of this paper.

THEOREM 3.1. *For any $\mathbf{f} \in H'$, problem (3.3) has a unique solution $(\mathbf{w}, \mathbf{u}) \in W \times H$ which continuously depends on data, i.e., there exists a constant $C = C(\alpha, \beta, \gamma) > 0$, such that*

$$\|\mathbf{w}\|_W + \|\mathbf{u}\|_X \leq C\|\mathbf{f}\|_{H'}$$

PROOF. The operator A is V-elliptic and the operator B satisfies the inf-sup conditions (3.5). Therefore, thanks to the result of [11,12], problem (3.3) has one and only one solution $(\mathbf{w}, \mathbf{u})$ which satisfies the *a priori* estimate (3.6). ∎

We shall next give another characterization of the solution of problem (3.3) which is well suited for the use of mixed finite-element methods. We are given an other Hilbert space $\tilde{W} := \mathbf{H}^1 \subset W$ and the operator $\tilde{B} : \tilde{W} \longrightarrow X'$ defined by

$$\langle B\mathbf{z}, \mathbf{v}\rangle_{H',H} = \langle \tilde{B}\mathbf{z}, \mathbf{v}\rangle_{X',X} = (\operatorname{grad}\mathbf{z}, \operatorname{grad}\mathbf{v})_{0,\Omega}, \qquad \mathbf{z} \in \tilde{W}, \quad \mathbf{v} \in H.$$

We have the following result.

LEMMA 3.4. INF-SUP CONDITION ON $\tilde{W}$. *There exists a constant $\tilde{\gamma} > 0$, such that*

$$\sup_{\mathbf{w}\in\tilde{W}} \frac{\left\langle \mathbf{v}, \tilde{B}\mathbf{w}\right\rangle_{X,X'}}{\|\mathbf{w}\|_{1,\Omega}} \geq \tilde{\gamma}\,\|\mathbf{v}\|_X, \qquad \forall\, \mathbf{v} \in H.$$

PROOF. Since $H \subset \tilde{W}$, we use the same proof as Lemma 3.3. ∎

We consider the following problem,

$$\begin{gathered}
\text{find } \mathbf{w} \in \tilde{W}, \mathbf{u} \in H \ \text{ such that,} \\
\langle A\mathbf{w}, \mathbf{z}\rangle_{\tilde{W}',\tilde{W}} - \langle \tilde{B}'\mathbf{u}, \mathbf{z}\rangle_{\tilde{W}',\tilde{W}} = 0, \qquad \forall\, \mathbf{z} \in \tilde{W}, \\
\langle \tilde{B}\mathbf{w}, \mathbf{v}\rangle_{X',X} = \frac{1}{\epsilon}\langle \mathbf{f}, \mathbf{v}\rangle_{H',H}, \qquad \forall\, \mathbf{v} \in H.
\end{gathered}$$

Since the bilinear form $\langle A\mathbf{w}, \mathbf{z}\rangle$ is not *a priori* $\operatorname{Ker}\tilde{B}$-elliptic, problem (3.9) is not well posed, at least in general. However, we have the following.

THEOREM 3.2. *Assume that the first argument* $\mathbf{w}$ *of the solution* $(\mathbf{w}, \mathbf{u})$ *of problem (3.3) belongs to* $\mathbf{H}^1$. *Then,* $(\mathbf{w}, \mathbf{u})$ *is the unique solution of problem (3.9).*

PROOF. Let $(\mathbf{w}, \mathbf{u})$ be a solution of problem (3.3). If $\mathbf{w}$ belongs to $\tilde{W}$, it follows from the properties (3.7) and (3.8) that $(\mathbf{w}, \mathbf{u})$ is a solution of problem (3.9). Hence, it remains only to show the uniqueness of the solution of (3.9).

Assume that $(\mathbf{w}, \mathbf{u}) \in \tilde{W} \times H$ satisfies

$$\langle A\mathbf{w}, \mathbf{z}\rangle_{\tilde{W}',\tilde{W}} - \langle \tilde{B}'\mathbf{u}, \mathbf{z}\rangle_{\tilde{W}',\tilde{W}} = 0, \qquad \forall \mathbf{z} \in \tilde{W},$$
$$\langle \tilde{B}\mathbf{w}, \mathbf{v}\rangle_{X',X} = 0, \qquad \forall \mathbf{v} \in H.$$

Taking $\mathbf{z} = \mathbf{w}$ in (3.10) and using (3.11), we get $0 = \langle A\mathbf{w}, \mathbf{w}\rangle_{\tilde{W}',\tilde{W}} = (\mathbf{w}, \mathbf{w})_{0,\Omega}$, so that $\mathbf{w} = 0$. Therefore, we obtain

$$0 = \langle \tilde{B}'\mathbf{u}, \mathbf{z}\rangle_{\tilde{W}',\tilde{W}} = \langle \tilde{B}\mathbf{z}, \mathbf{u}\rangle_{X',X},$$

which implies $\mathbf{u} = 0$ by using the inf-sup condition (3.8). ∎

For the nonlinear case, the mixed variational formulation of (1.1) with the presence of the pressure term can be stated as follows, given $\mathbf{f} \in \mathbf{H}^{-1}$, find $\mathbf{w} \in \mathbf{H}^1, \mathbf{u} \in \mathbf{H}_0^1(\Omega), p \in L_0^2(\Omega)$ such that,

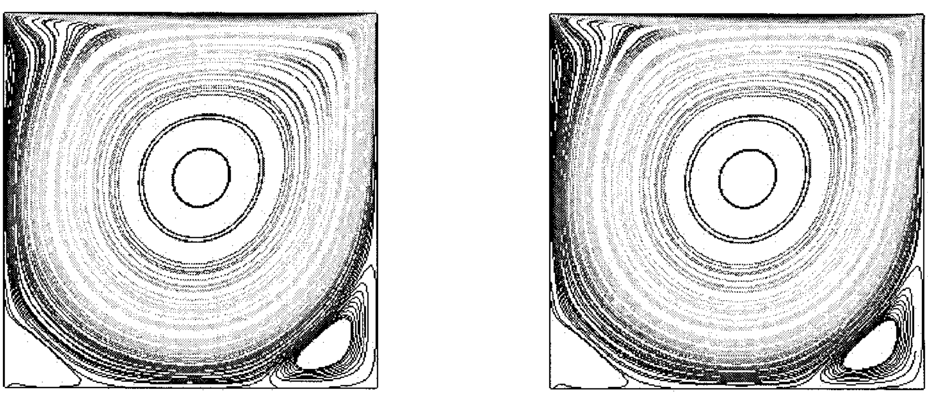
$$\int_\Omega \mathbf{w} \cdot \theta \, dx + \int_\Omega \nabla \mathbf{u} \cdot \nabla \theta \, dx = 0, \qquad \forall \theta \in \mathbf{H}^1,$$
$$\epsilon \int_\Omega \nabla \mathbf{w} \cdot \nabla \mathbf{v} \, dx - \nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx - \int_\Omega (\mathbf{u}\,\nabla)\mathbf{u} \cdot \mathbf{v} \, dx,$$
$$+ \int_\Omega p \nabla \cdot \mathbf{v} \, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx, \qquad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$
$$\int_\Omega q \nabla \cdot \mathbf{u} \, dx = 0, \qquad \forall q \in L_0^2(\Omega).$$

Problem (3.12) is easier to approximate numerically than problem (1.1) since the finite-element approximation of (3.21) involves only the construction of finite-dimensional subspaces of the spaces $\mathbf{H}^1$, $\mathbf{H}_0^1(\Omega)$, $L_0^2(\Omega)$.

The analysis of the mixed formulation for the nonlinear case and the time dependent case will be studied elsewhere.

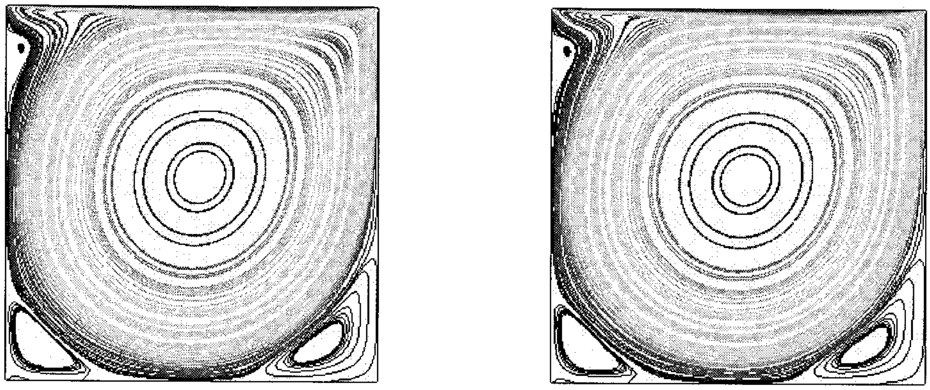
4. NUMERICAL RESULTS

In this section, we perform some numerical tests on the equations (3.12).



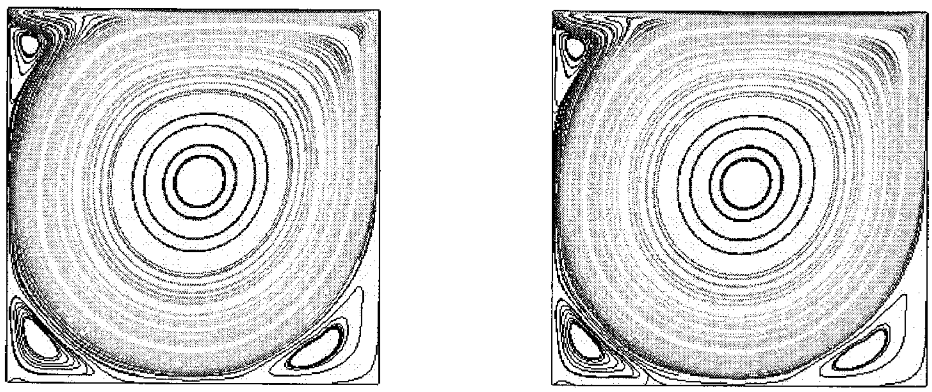
(a) (b)

Figure 1. Streamline velocity field for Re = 1200 and for $\epsilon = 0$ and $\epsilon = 10^{-9}$, respectively.



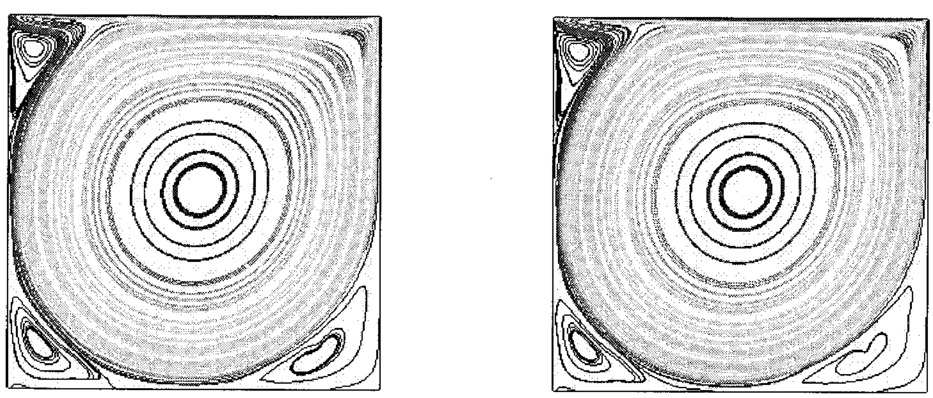
(a) (b)

Figure 2. Streamline velocity field for Re = 2400 and for $\epsilon = 0$ and $\epsilon = 10^{-9}$, respectively.



(a) (b)

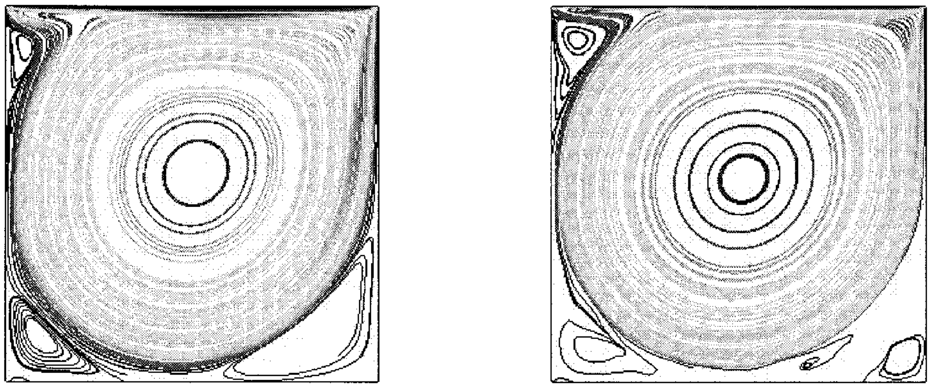
Figure 3. Streamline velocity field for Re = 4000 and for $\epsilon = 0$ and $\epsilon = 10^{-9}$, respectively.



(a) (b)

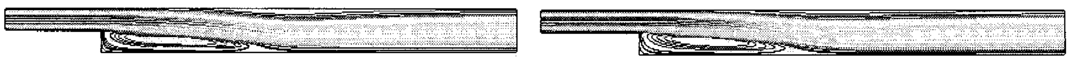
Figure 4. Streamline velocity field for Re = 6000 and for $\epsilon = 0$ and $\epsilon = 10^{-9}$, respectively.

To approximate the velocity and the pressure $(\mathbf{u}, p)$, we have used the $P_2 - P_1$ standard Lagrange finite element, while the function $\mathbf{w}$ is approximated by the P_2 Lagrange element.



(a) (b)

Figure 5. **u** streamlines for (Re = 6000, $\epsilon = 10^{-7}$) and (Re = 9230, $\epsilon = 10^{-9}$), respectively.



(a) (b)

Figure 6. Streamline velocity field for Re = 300 and for $\epsilon = 0$ and $\epsilon = 10^{-6}$, respectively.

(a) (b)

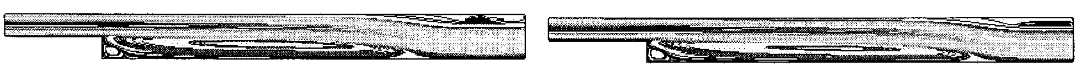
Figure 7. Streamline velocity field for Re = 600 and for $\epsilon = 0$ and $\epsilon = 10^{-6}$, respectively.

(a) (b)

Figure 8. Streamline velocity field for Re = 750 and for $\epsilon = 0$ and $\epsilon = 10^{-6}$, respectively.

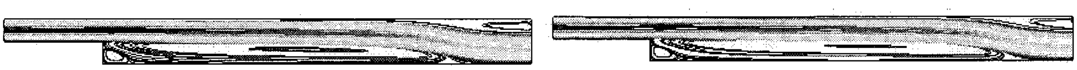
(a) (b)

Figure 9. Streamline velocity field for Re = 860 and for $\epsilon = 0$ and $\epsilon = 10^{-6}$, respectively.



(a) (b)

Figure 10. **u** streamlines for (Re = 1430, $\epsilon = 10^{-6}$) and (Re = 1765, $\epsilon = 10^{-6}$), respectively.



(a) (b)

Figure 11. **u** streamlines for (Re = 2300, $\epsilon = 10^{-6}$) and (Re = 3000, $\epsilon = 10^{-6}$), respectively.

The resulting nonlinear system is solved by using the Femlab library.

The performance of the numerical method described above has been tested on two problems for different parameter ϵ and different Reynold numbers $\mathrm{Re} = (UL)/\nu$ where U and L are some characteristic constants.

A widely used test cases for benchmarking incompressible flow codes are represented by the lid-driven cavity and the backward facing step flows.

LID-DRIVEN CAVITY PROBLEM. The fluid contained inside a $[0,1] \times [0,1]$ square cavity is set into motion by the upper wall which is sliding at constant speed. The boundary conditions are: $(u,v) = (1,0)$ at the top, and $(u,v) = (0,0)$ on the other parts of the boundary. We have used for all our simulations a uniform grid mesh of 7000 elements. We performed computations for Reynolds numbers ranging between 1200 and 10^4. The streamline velocity fields are presented in Figures 1–5. The computations were performed for different Reynolds numbers and ϵ. For $\mathrm{Re} \leq 6000$ and $\epsilon = 10^{-9}$, we see from Figures 1–4 that the solution pratically coincides with the Navier-Stokes one ($\epsilon = 0$). In Figure 5, we varied the value of ϵ from 10^{-9} to 10^{-7} in order to observe the primary and the secondary vortices at the bottom right corner of the cavity. These results are in good agrement with the literature [13]. For $\mathrm{Re} \geq 9000$, we still have convergence for $\epsilon = 10^{-9}$ and for $\epsilon = 0$ the solution do not converge.

BACKWARD FACING STEP. We solve for a flow in a plane channel with sudden step expansion. Instead, where the geometry changes abruptly, the flow separate and develops a recirculation region. The ration of a step height H to the channel outlet width L is $H/L = 1/2$. We have used the following domain geometry: $H = 0.2, L = 0.4$, total length $L_T = 5$, and the length from the inlet to the step $L_s = 1$. While the boundary conditions are: at the inlet the profile of the velocity is parabolic with $U_{\max} = 1.5$, at the outlet $v = 0$, and on the other parts of the boundary $(u,v) = (0,0)$. The streamline velocity fields are presented in Figures 6–11. The computations were performed for different Reynolds numbers and for different parameter ϵ. We performed computations for Reynolds numbers ranging between 300 and 3000. For $\mathrm{Re} \leq 500$ and $\epsilon = 10^{-6}$, we see from Figure 6 that the solution pratically coincides with the Navier-Stokes one ($\epsilon = 0$). In Figures 7–11, we note, however, a presence of secondary vortice at the bottom left corner of the step in the case where $\epsilon \neq 0$. For $\mathrm{Re} \geq 1000$, we still have convergence for $\epsilon = 10^{-6}$ and for $\epsilon = 0$ the solution do not converge. For the Reynolds number $\mathrm{Re} = 3000$, the calculated reattachment length is $x_r/H \approx 9$.

In conclusion, we have to point out the dissipative role of the parameter ϵ in our numerical simulations.

REFERENCES

1. J. Necas and M. Silhavy, Multipolar viscous fluids, *Quart. Appl. Math.* **49**, 247–265, (1991).
2. A.E. Green and R.S. Rivlin, Simple force and stress multipoles, *Arch. Rat. Mech. Anal.* **17**, 325–353, (1964).
3. A. Brahmi, *Mémoire de maîtrise*, Université Laval, Québec, (1994).
4. J. Malek and S. Turek, Non-Newtonian flow prediction by divergence free finite elements, *Stability Appl. Anal. Contin. Media.* **3**, 165–180, (1993).
5. K.K. Golovkin, New equations modeling the motion of a viscous fluid, and their unique solvability, *Proc. Steklov Inst. Math.* **102**, (1967).
6. O.A. Ladyzhenkaya, New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems, *Trudy Mat. Inst. Steklov.* **102**, 95–118, (1967).
7. J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, (1969).
8. M. Cannone and Grzegorz, *About the Regularized Navier-Stokes Equations*, (Preprint), (2003).
9. Y.U. Ou and S.S. Sritharan, Analysis of regularized Navier-Stokes equations I, *Quart. Appl. Math.* **49**, 651–685, (1991).
10. V. Girault and P.A. Raviart, *Finite Element Approximation of Navier-Stokes Equations*, Lecture in Notes in Mathematics, pp. 749, Springer-Verlag, Berlin, (1979).
11. A. Quarteroni, R. Sacco and F. Saleri, *Numerical Mathematics*, Springer, (2000).
12. C. Amrouche and V. Girault, Problèmes de Stokes généralisés, *Portugaliae Mathematica* **49**, (1992).
13. J. Kim and P. Moin, Application of a fractional-step method to incompressible Navier-Stokes equations, *J. Comp. Physics.* **59**, 308–323, (1985).